

Linear progress in the complex of curves

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Abstract

We show that a random walk on the mapping class group of an orientable surface of finite type makes linear progress in the relative metric, which is quasi-isometric to the complex of curves.

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1 Introduction

Let Σ be an orientable surface of finite type, which is not a sphere with three or fewer punctures. The mapping class group G of Σ is the group of orientation preserving diffeomorphisms of Σ , modulo those isotopic to the identity. Let μ be a probability distribution on G . We may use μ to generate a random walk on G , which is a Markov chain on G with transition probabilities $p(x, y) = \mu(x^{-1}y)$, and we will assume we start at the identity at time zero. The path space for the random walk is the probability space $(G^{\mathbb{Z}_+}, \mathbb{P})$, where the product $G^{\mathbb{Z}_+}$ is the collection of all sample paths, and the measure \mathbb{P} is determined by μ . Let w_n be the random variable corresponding to projection onto the n -th factor. So if ω is a sample path, $w_n(\omega)$ is the location of the path at time n , and the distribution of w_n is given by the n -fold convolution of μ with itself, which we shall write μ_n . We

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say that the random walk has a linear rate of escape if the limit $\ell = \lim_{n \rightarrow \infty} \frac{1}{n} |w_n(\omega)|$ exists for almost all sample paths ω , and ℓ is strictly greater than zero almost surely. Here $|g|$ is the length of the group element g in the word metric on the group G . Kesten [7, 8] and Day [2] showed that an irreducible random walk on a non-amenable group has a linear rate of escape, assuming the random walk has finite first moment, i.e. the expected value of the distance the random walk moves in one step is finite. The mapping class group contains non-abelian free subgroups, so in particular is non-amenable. Therefore a random walk on the mapping class group makes linear progress in the word metric on the group.

Masur and Minsky [12] show that there is a relative metric on the mapping class group, under which the group is quasi-isometric to the complex of curves. A relative metric is a word metric on the group with respect to an infinite generating set, consisting of a finite generating set, union a finite collection of subgroups. In this case, the collection of subgroups consists of stabilizers of simple closed curves α_i , where the α_i consist of representatives for orbits of simple closed curves under the action of the mapping class group. The complex of curves is a simplicial complex whose vertices are isotopy classes of essential simple closed curves, and whose simplices are spanned by disjoint collections of simple closed curves. The quasi-isometry may be explicitly described by choosing a basepoint x_0 for the complex of curves, and sending g to $g(x_0)$. We will write $\hat{d}(1, g)$ for the length of the group element g in the relative metric on G , and this is coarsely equivalent to the distance g moves the basepoint in the complex of curves. In this paper we show that a random walk in the mapping class group makes linear progress in the relative metric.

Theorem 1.1. *Let G be the mapping class group of an orientable surface of finite type, which is not a sphere with three or fewer punctures, and consider the random walk generated by a probability distribution μ whose support is bounded in the relative metric, which generates a non-elementary subgroup of the mapping class group and which has finite first moment. Then there is a constant $\ell > 0$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \hat{d}(1, w_n) = \ell$ almost surely.*

The relative metric is an improper metric on the mapping class group, and therefore not quasi-isometric to the word metric, so linear progress in the word metric does not immediately imply linear progress in the relative metric. We will begin with a brief discussion of the mapping class group of the torus, as in this case it is easy to see that the result holds.

The mapping class group of the torus is isomorphic to the group $SL(2, \mathbb{Z})$. As there are no non-parallel disjoint essential simple closed curves on the torus, the curve complex for the torus is usually defined to have edges connecting pairs of curves that intersect exactly once. The curve complex for the torus is the Farey triangulation of the unit disc, and $SL(2, \mathbb{Z})$ with its word metric is quasi-isometric to the dual graph, which is a trivalent tree. This is illustrated below in Figure 1.

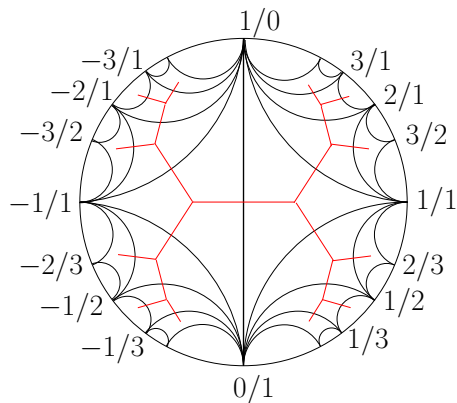


Figure 1: The Farey triangulation.

For simplicity, consider a non-backtracking random walk on the trivalent tree. Such a path may be described by a sequence $\text{LRLRL}\dots$, where an L denotes a left turn at a vertex, and an R denotes a right turn at a vertex. Such a path makes uniform progress in the trivalent graph. The path travels distance one in the relative space whenever the next letter in the sequence is different from the previous one, and this occurs with probability one-half. So the random walk makes progress in the relative space on average at half the rate it makes progress in the trivalent tree.

We now indicate the argument we use for more complicated mapping class groups. Imagine starting at a basepoint x_0 in hyperbolic space, and travelling some large distance R . The set of points you may arrive at is given by a sphere centered at x_0 . Now travel another large distance S , as illustrated below in Figure 2.

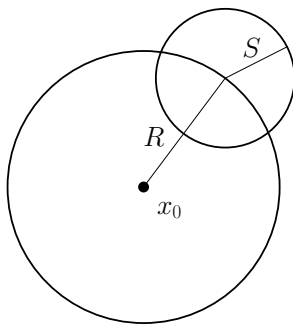


Figure 2: Spheres in hyperbolic space.

Due to negative curvature, nearly all the volume of the second sphere of radius S lies outside the sphere of radius R . We wish to translate this intuition into our setting. Our space is the mapping class group G with a relative metric, which we shall denote \widehat{G} , which is quasi-isometric to the complex of curves. Masur and Minsky [12] showed that this space is a (non-proper) δ -hyperbolic space, and Klarreich [10] identified the Gromov boundary of the space as the space of foliations in \mathcal{PMF} which

contain no closed trajectories. Let μ be a probability distribution on G , whose support generates a non-elementary subgroup. Kaimanovich and Masur showed that for the random walk determined by μ , almost all sample paths converge to uniquely ergodic, and hence minimal, foliations and this defines a harmonic measure ν on the boundary, where $\nu(X)$ is the probability that a sample path converges to a foliation contained in the set X . The harmonic measure ν governs the long time behaviour of the sample paths, and is the weak- \star limit of the n -fold convolutions of μ on \widehat{G} union its boundary. In particular, if we start at the identity and consider all sample paths of length R , for R large, the distribution of endpoints looks similar to ν , at least when viewed from the identity. If we now continue the random walk for another S steps, negative curvature leads us to expect that most of the new sample points will lie a definite distance further away from the origin.

In order to make this intuition precise, we need some way of comparing the harmonic measure ν with the convolution measures μ_n . We will compare the measures on sets which are *halfspaces*. A halfspace $H(x, y)$ consists of all points closer (in \widehat{G}) to y than x , and we will be most interested in halfspaces $H(1, x)$, where 1 is the identity element in G . In Section 5 we will show that the harmonic measure of a halfspace $H(1, x)$ decays exponentially in $\widehat{d}(1, x)$, i.e. there is a constant $L < 1$ such that $\nu(\overline{H(1, x)}) \leq L^{\widehat{d}(1, x)}$, for all x sufficiently far from 1, where $\overline{H(1, x)}$ is the closure of $H(1, x)$. Furthermore, we will show there is a constant Q such that $\mu_n(H(1, x)) \leq QL^{\widehat{d}(1, x)}$, for all x sufficiently far from 1, and these estimates will allow us to relate the harmonic and convolution measures. We briefly indicate why one expects these estimates to hold. First observe that there is some number K such that the harmonic measure of all halfspaces $H(1, x)$, with $\widehat{d}(1, x) \geq K$ is bounded away from 1, at most $1 - \epsilon$, say. Then given a halfspace $H(1, x_1)$, where $\widehat{d}(1, x_1)$ is large, one may construct a nested sequence of half spaces $H(1, x_n) \supset H(1, x_{n-1}) \supset \dots \supset H(1, x_1)$. The number of such halfspaces is linear in $\widehat{d}(1, x_i)$, and furthermore, we may assume that the distance between any point in $H(1, x_i)$ and any point in the complement of $H(1, x_{i+1})$ is larger than K . The conditional probability that a sample path converges into $H(1, x_i)$, given that it hits a point in the complement of $H(1, x_{i+1})$ is at most $1 - \epsilon$, so the harmonic measure of the innermost halfspace is at most $(1 - \epsilon)^n$. In order to bound μ_n in terms of ν , suppose a large amount of the mass of μ_n is contained in $H(1, x_i)$, then at most $(1 - \epsilon)\mu_n(H(1, x_i))$ of the mass can escape back out in to the complement of $H(1, x_{i+1})$, so this gives the upper bound for μ_n .

We will use the estimates described above to find a positive lower bound on the expected extra distance from 1 obtained by taking an extra m steps, after a random walk of length n , for m sufficiently large, i.e. we will show $\mathbb{E}(\widehat{d}(1, w_{n+m}) - \widehat{d}(1, w_n)) \geq \delta > 0$. Therefore $\mathbb{E}(\widehat{d}(1, w_{km})) \geq k\delta$, so the expected distance from the identity grows linearly, and a standard application of Kingman's subadditive ergodic theorem [9] shows that this will then be true for almost all sample paths as well.

We remark that there are distance non-increasing maps to the relative space from other useful spaces on which the mapping class group acts, such as Teichmüller space and the pants complex, so linear progress in the complex of curves implies linear progress in these spaces too. In particular, work of Duchin [3], shows that linear progress in Teichmüller space implies that for almost all sample paths there is a geodesic which the random walk tracks sublinearly, at least for the parts of the geodesic in the thick part of Teichmüller space.

In Section 2, we recall some standard definitions and set up some notation. In Section 3 we prove some useful results about halfspaces in non-proper δ -hyperbolic metric spaces. As we do not assume that μ is symmetric, it will be convenient for us to know that a semi-group in the mapping class group contains a pair of independent pseudo-Anosov elements, if and only if it generates a non-elementary subgroup, and we show this in Section 4, in a straight forward extension of some results

of Ivanov [5]. In Section 5 we show that the harmonic measure of halfspaces in the relative space decays exponentially in the distance of the halfspace from the basepoint, and obtain the estimate for the convolution measures. Finally in Section 6 we find a positive lower bound for the expected difference between $\hat{d}(1, w_{n+m})$ and $\hat{d}(1, w_n)$, and then apply Kingman's subadditive ergodic theorem to show that a random walk makes linear progress in the relative space.

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2 Preliminaries

Let Σ be an orientable surface of finite type, i.e. a surface of genus g with p marked points, usually referred to as punctures. The mapping class group G of Σ consists of orientation preserving diffeomorphisms of Σ which preserve the punctures, modulo those isotopic to the identity. For the purposes of this paper we shall assume that Σ is not a sphere with three or fewer punctures.

The collection of essential simple closed curves in the surface may be made into a simplicial complex, called the *complex of curves*, which we shall denote $\mathcal{C}(\Sigma)$. The vertices of this complex are isotopy classes of simple closed curves in Σ , and a collection of vertices spans a simplex if representatives of the curves can be realised disjointly in the surface. The complex of curves is a finite dimensional simplicial complex, but it is not locally finite. We will write $\mathcal{C}_0(\Sigma)$ to denote the vertices of the simplicial complex $\mathcal{C}(\Sigma)$, which is the set of isotopy classes of simple closed curves. We will write $d_{\mathcal{C}}(x, y)$ for the distance in the one-skeleton between two vertices x and y of the complex of curves. We will always consider the complex of curves to have a basepoint x_0 , which we can take to be one of the curves corresponding to a standard generating set for the mapping class group. The mapping class group acts by simplicial isometries on the complex of curves. For certain sporadic surfaces the definition above produces a collection of disconnected points, and so a slightly different definition is used. If the surface is a torus with at most one puncture, then two vertices are connected by an edge if the corresponding simple closed curves may be isotoped to intersect transversely exactly once. If the surface is a four punctured sphere, then two vertices are connected by an edge if the corresponding simple closed curves may be isotoped to intersect transversely in two points. In both of these cases, the resulting curve complex is isomorphic to the Farey graph.

A geodesic metric space is δ -hyperbolic if every geodesic triangle is δ -slim, i.e. each edge is contained in a δ -neighbourhood of the other two. Masur and Minsky [12] have shown that the complex of curves is δ -hyperbolic.

The mapping class group is finitely generated, so any choice of generating set A gives rise to a word metric on G , and any two different choices of finite generating set give quasi-isometric word metrics. Given a group G , and a collection of subgroups $\mathcal{H} = \{H_i\}_{i \in I}$, we define the *relative length* of a group element g to be the length of the shortest word in the typically infinite generating set $A \cup \mathcal{H}$. This defines a metric on G called the *relative metric*, which depends on the choice of subgroups \mathcal{H} . We will write \hat{G} to denote the group G with the relative metric, which we shall also refer to as the *relative space*. We say a finitely generated group G is *weakly relatively hyperbolic*, relative to a finite list of subgroups \mathcal{H} , if the relative space \hat{G} is δ -hyperbolic.

We may consider the relative metric on the mapping class group with respect to the following collection of subgroups. Let $\{\alpha_1, \dots, \alpha_n\}$ be a list of representatives of orbits of simple closed curves in Σ , under the action of the mapping class group. Let $H_i = \text{fix}(\alpha_i)$ be the subgroup of G fixing α_i . Masur and Minsky [12] have shown that the resulting relative space is quasi-isometric to the complex of curves. As the complex of curves is δ -hyperbolic, this shows that the mapping class group is weakly relatively hyperbolic. Klarreich [10], see also Hamenstädt [4], showed that the Gromov boundary of the complex of curves is the space \mathcal{F}_{\min} , which we now describe. The space \mathcal{F}_{\min} consists of the subset of \mathcal{PMF} consisting of foliations which contain no closed trajectories, up to equivalence, where two measured foliations are equivalent if they are topologically equivalent. The topology on \mathcal{F}_{\min} is the induced topology from \mathcal{PMF} .

We now review some background on random walks on groups, see for example Woess [14]. Let G be the mapping class group of an orientable surface of finite type, which is not a sphere with three or fewer punctures, and let μ be a probability distribution on G . We may use the probability distribution μ to generate a Markov chain, or *random walk* on G , with transition probabilities $p(x, y) = \mu(x^{-1}y)$. We shall always assume that we start at time zero at the identity element of the group. The *path space* for the random walk is the probability space $(G^{\mathbb{Z}^+}, \mathbb{P})$, where $G^{\mathbb{Z}^+}$ is the set of all infinite sequences of elements G . We will write w_n for the random variable corresponding to projection onto the n -th factor, which gives the position of the sample path at time n . The position of the random walk at time n may be described as the product $m_1 m_2 \dots m_n$, where the m_i are the *increments* of the random walk, i.e. the m_i are a sequence of independent μ -distributed random variables. Therefore the distribution of random walks at time n is given by the n -fold convolution of μ , which we shall write as μ_n , and we shall write $p^{(n)}(x, y)$ for the probability that you go from x to y in n steps. The probability measure \mathbb{P} is determined by μ_n using the Kolmogorov extension theorem. The *reflected* random walk is the walk generated by the reflected measure $\tilde{\mu}$, where $\tilde{\mu}(g) = \mu(g^{-1})$. The *Bernoulli shift* in the space of increments of the random walk determines a measure-preserving ergodic transformation on $(G^{\mathbb{Z}^+}, \mathbb{P})$ determined by $(Uw)_n = w_1^{-1}w_{n+1}$.

We shall always require that the group generated by the support of μ is *non-elementary*, which means that it contains a pair of pseudo-Anosov elements with distinct fixed points in \mathcal{PMF} , the space of projective measured foliations on the surface. We do not assume that the probability distribution μ is symmetric, so the group generated by the support of μ may be strictly larger than the semi-group generated by the support of μ . Finally, we shall always assume that the probability distribution μ has finite first moment with respect to the word metric on G .

In [11], we showed that it followed from results of Kaimanovich and Masur [6] and Klarreich [10], that a sample path converges almost surely to a uniquely ergodic foliation in the Gromov boundary of the relative space. This gives a measure ν on \mathcal{F}_{\min} , known as harmonic measure. The harmonic measure ν is μ -stationary, i.e.

$$\nu(X) = \sum_{g \in G} \mu(g) \nu(g^{-1}X).$$

Theorem 2.1. [6, 10, 11] *Consider a random walk on the mapping class group of an orientable surface of finite type, which is not a sphere with three or fewer punctures, determined by a probability distribution μ such that the group generated by the support of μ is non-elementary. Then a sample path $\{w_n(\omega)\}$ converges to a uniquely ergodic foliation in the Gromov boundary \mathcal{F}_{\min} of the relative space \hat{G} almost surely, and the distribution of limit points on the boundary is given by a unique μ -stationary, non-atomic measure ν on \mathcal{F}_{\min} .*

We remark that the measure ν is supported on the uniquely ergodic foliations, which are a subset

of \mathcal{PMF} , so we may think of ν as a measure on \mathcal{PMF} , with zero weight on all the non-uniquely ergodic measures.

3 Halfspaces

In this section we give detailed proofs of various useful properties of halfspaces in a non-locally compact δ -hyperbolic space. These properties are presumably well known for locally compact δ -hyperbolic spaces, but we provide complete proofs to verify that these properties hold in the non-locally compact case. For consistency with the other sections of this paper, we will denote our non-locally compact δ -hyperbolic metric space by \widehat{G} , and we will write $\widehat{d}(x, y)$ for the distance in \widehat{G} between two points x and y . Furthermore, we will always have a distinguished base point, which we shall call 1.

Two points a and b in a metric space define a *halfspace* $H(a, b)$ consisting of all those points which are at least as close to b as to a , i.e. $H(a, b) = \{x \in \widehat{G} \mid \widehat{d}(x, b) \leq \widehat{d}(x, a)\}$. The main two results of this section are Propositions 3.9 and 3.10, which we now briefly describe. Proposition 3.9 says if two halfspaces $H(1, x)$ and $H(1, y)$ are small and far apart, then any other small halfspace hits at most one of $H(1, x)$ or $H(1, y)$. The halfspaces $H(1, x)$ and $H(1, y)$ are small if $\widehat{d}(1, x)$ and $\widehat{d}(1, y)$ are large, and they are far apart if the geodesic $[x, y]$ passes close to the basepoint 1. In this case any other halfspace $H(1, z)$ hits at most one of $H(1, x)$ or $H(1, y)$, as long as $\widehat{d}(1, z)$ is sufficiently large. Proposition 3.10 says that given a halfspace $H(1, x)$, we may choose a point y on a geodesic $[1, x]$ close to x such that the halfspace $H(1, x)$ is contained in the halfspace $H(1, y)$. We also obtain specific bounds on how large the halfspace $H(1, x)$ appears when viewed from any point in $H(y, 1)$, and similar bounds on how large the halfspace $H(y, 1)$ appears when viewed from any point in $H(1, x)$. To be more precise, we show that for any point a in $H(y, 1)$, the halfspace $H(1, x)$ is contained in a halfspace $H(a, b)$, with an explicit lower bound on $\widehat{d}(a, b)$. Furthermore, for any point b in $H(1, x)$, there is a halfspace $H(b, a)$ such that $H(y, 1) \subset H(b, a)$, again, with an explicit lower bound on $\widehat{d}(a, b)$.

We begin with some elementary observations about nearest point projections. In a δ -hyperbolic space nearest point projections onto quasi-convex sets are coarsely well defined. We now show that for any set X , the K -neighbourhoods of X , and a particular choice of shortest path to X , have bounded intersection.

Proposition 3.1. *Let X be a set, and let $[z, p]$ be a minimal length geodesic from a point z to X . Then the intersection of the K -neighbourhoods of X and $[z, p]$ is contained in a $3K$ -neighbourhood of p .*

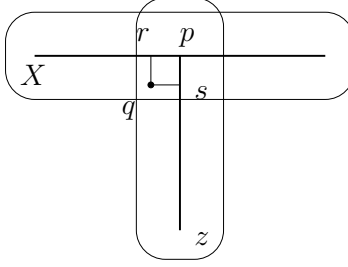


Figure 3: A shortest path from z to X .

Proof. Choose a point q in the intersection of $N_K(X)$ and $N_K([z, p])$. There are points $r \in X$ and $s \in [z, p]$ such that $\widehat{d}(r, q) \leq K$ and $\widehat{d}(s, q) \leq K$, as illustrated above in Figure 3. If $\widehat{d}(p, q) > 3K$, then as $\widehat{d}(s, q) \leq K$, this implies that $\widehat{d}(p, s) > 2K$. Therefore r is a closer point on $[x, y]$ to s , and hence to z , than p , which contradicts the fact that p is a closest point on X to z . So $\widehat{d}(p, q) \leq 3K$, and so $N_K(X) \cap N_K([z, p]) \subset N_{3K}(p)$, as required. \square

This shows that, up to additive error, the shortest way to get from a point z to a point x on a geodesic, is to head to the closest point to z on the geodesic, and then run along the geodesic to x . We will make extensive use of this fact, so we record it here as a proposition.

Proposition 3.2. *Let $[x, y]$ be a geodesic from x to y , and p a closest point on $[x, y]$ to z . Any geodesic $[x, z]$ from x to z intersects a 3δ -neighbourhood of p , so $[x, p] \cup [p, z]$ is contained in a 3δ -neighbourhood of $[x, z]$. In particular $\widehat{d}(x, p) + \widehat{d}(p, z) - 6\delta \leq \widehat{d}(x, z) \leq \widehat{d}(x, p) + \widehat{d}(p, z)$.*

Proof. The right hand inequality is just the triangle inequality. We now justify the left hand inequality. By thin triangles, any geodesic $[x, z]$ from x to z is contained in a δ -neighbourhood of the union of geodesics $[x, p]$ and $[p, z]$. If $[p, x]$ and $[p, z]$ had long initial segments that fellow travelled, then the distance from x to z might be much shorter than the sum of the distances from x to p and from z to p . However, this would contradict the fact that p was a closest point on $[x, y]$ to z . To be precise, Proposition 3.1 implies that the intersection of the δ -neighbourhoods of $[x, p]$ and $[p, z]$ is contained in a 3δ -neighbourhood of p . This means that any geodesic from z to x must pass within 3δ of p , so $\widehat{d}(z, p) + \widehat{d}(p, x) - 6\delta \leq \widehat{d}(z, x)$, as required. \square

An immediate consequence of Proposition 3.2 is the following quantitative version of the fact that nearest point projection is coarsely well defined.

Proposition 3.3. *Let p and q be nearest points to z on a geodesic $[x, y]$. Then $\widehat{d}(p, q) \leq 6\delta$.*

Let a and b have nearest point projections p and q onto a geodesic $[x, y]$. We now show that if $\widehat{d}(p, q) > 14\delta$ apart, then, up to additive error, the geodesic from a to b goes from a to p , then runs along the geodesic from p to q , and then heads back out to b .

Proposition 3.4. *Let $[x, y]$ be a geodesic and let p be a closest point on $[x, y]$ to a , and let q be a closest point on $[x, y]$ to b . If $\widehat{d}(p, q) > 14\delta$ then $\widehat{d}(a, b) \geq \widehat{d}(a, p) + \widehat{d}(p, q) + \widehat{d}(q, b) - 24\delta$.*

Proof. Let $[x, y]$ be a geodesic from x to y . Let $[a, p]$ be a minimal length geodesic from a to $[x, y]$, and let $[b, q]$ be a minimal length geodesic from b to $[x, y]$. First we show that $N_{2\delta}([a, p])$ and $N_{2\delta}([b, q])$ are disjoint. Suppose not, then let r be a point in $N_{2\delta}([a, p]) \cap N_{2\delta}([b, q])$. Then there are points $s \in [a, p]$ and $t \in [b, q]$ such that $\widehat{d}(s, r) \leq 2\delta$ and $\widehat{d}(t, r) \leq 2\delta$, as illustrated in Figure 4 below.

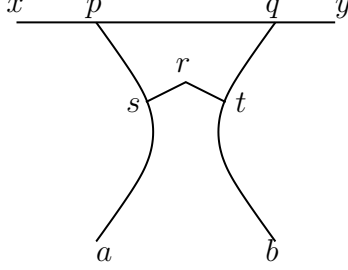


Figure 4: Shortest paths from a and b to $[x, y]$.

By the triangle inequality, $\widehat{d}(t, p) \leq \widehat{d}(t, s) + \widehat{d}(s, p)$. As $\widehat{d}(t, s) \leq 4\delta$, this shows that

$$\widehat{d}(t, p) \leq 4\delta + \widehat{d}(s, p). \quad (3.1)$$

By Proposition 3.2, the path from t to p via q is almost a geodesic, i.e.

$$\widehat{d}(t, q) + \widehat{d}(q, p) - 6\delta \leq \widehat{d}(t, p). \quad (3.2)$$

Combining (3.1) and (3.2), we obtain

$$\widehat{d}(t, q) + \widehat{d}(q, p) - 6\delta \leq \widehat{d}(s, p) + 4\delta. \quad (3.3)$$

As $[a, p]$ is a minimal length geodesic from a to $[x, y]$, it is also a minimal length geodesic from any point on $[a, p]$ to $[x, y]$. Therefore p is a closest point on $[x, y]$ to s , so the distance from s to p is less than or equal to the distance from s to $[x, y]$ by a path through t . This implies that

$$\widehat{d}(s, p) \leq 4\delta + \widehat{d}(t, q). \quad (3.4)$$

Combining (3.3) and (3.4), and subtracting $\widehat{d}(t, q)$ from both sides, implies that $\widehat{d}(p, q) \leq 14\delta$. However we assumed that $\widehat{d}(p, q) > 14\delta$, so this implies that $N_{2\delta}([a, p])$ and $N_{2\delta}([b, q])$ are in fact disjoint.

By thin triangles, any geodesic $[a, b]$ is contained in 2δ -neighbourhood of $[a, p] \cup [p, q] \cup [q, b]$. By Proposition 3.1, the intersection of $N_{2\delta}([a, p])$ with $N_{2\delta}([x, y])$ is contained in a 6δ -neighbourhood of p . Similarly, the intersection of $N_{2\delta}([b, q])$ with $N_{2\delta}([x, y])$ is contained in a 6δ -neighbourhood of q . As the remaining parts of the 2δ -neighbourhoods outside $N_{6\delta}(p)$ and $N_{6\delta}(q)$ are disjoint, this means that $\widehat{d}(a, b) \geq \widehat{d}(a, p) + \widehat{d}(p, q) + \widehat{d}(q, b) - 24\delta$, as required. \square

We now show that nearest point projection to a connected subgeodesic of a geodesic is coarsely equivalent to nearest point projection to the original geodesic, followed by nearest point projection to the subgeodesic.

Proposition 3.5. *Let ρ_1 be nearest point projection onto a geodesic $[a, b]$, and let ρ_2 be nearest point projection onto a subgeodesic $[c, d] \subset [a, b]$. Then there is a constant K_1 , which only depends on δ , such that $\widehat{d}(\rho_2(x), \rho_2(\rho_1(x))) \leq K_1$, for any point x .*

Proof. Let p be the nearest point projection of x to $[a, b]$, i.e. $p = \rho_1(x)$. If q is another point on $[a, b]$, then the path from x to q via p is almost a geodesic, Proposition 3.2, so $\widehat{d}(x, p) + \widehat{d}(p, q) - 6\delta \leq \widehat{d}(x, q) \leq \widehat{d}(x, p) + \widehat{d}(p, q)$. Therefore, if p lies in the subgeodesic $[c, d]$, then $\pi_2(p) = p$, and any point in $[c, d]$ further than 6δ from p is further away from x than p . On the other hand, if p lies outside $[c, d]$, then $\rho_2(p)$ is equal to one of the endpoints, which we may assume is c , up to relabelling. Again, Proposition 3.4 implies that any point on $[c, d]$ more than 6δ away from c is further away from x than c . So we may choose K_1 to be 7δ . \square

We now show that that if a path γ lies in a bounded neighbourhood of a geodesic, then the nearest point projections of any point to the path and to the geodesic are a bounded distance apart.

Proposition 3.6. *Let $[x, y]$ be a geodesic, and let γ be any path from x to y contained in a K -neighbourhood of $[x, y]$. For any point z , let p be a closest point on $[x, y]$ to z , and let q be a closest point on γ to z . Then $\widehat{d}(p, q) \leq 3K + 6\delta$.*

Proof. Let p be the closest point to z on $[x, y]$. As γ is a path from $[x, y]$ contained in a K -neighbourhood of $[x, y]$, there is a point p' in γ such that $\widehat{d}(p, p') \leq K$, and hence $\widehat{d}(p', z) \leq \widehat{d}(z, p) + K$. Let q be the closest point on γ to z , and let q' be the closest point on $[x, y]$ to q , so the distance from $\widehat{d}(q, q') \leq K$. This is illustrated below in Figure 5.

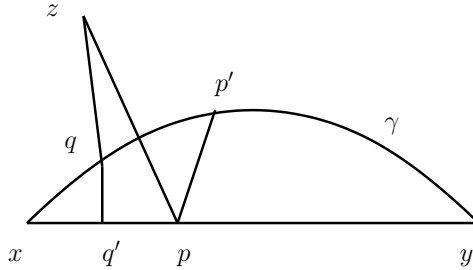


Figure 5: A path close to a geodesic $[x, y]$.

By Proposition 3.2, the path from z to q' via p is almost a geodesic, so $\widehat{d}(z, q') \geq \widehat{d}(z, p) + \widehat{d}(p, q') - 6\delta$, which implies that $\widehat{d}(z, q) \geq \widehat{d}(z, p) + \widehat{d}(p, q') - 6\delta - K$. Therefore if $\widehat{d}(p, q') > 2K + 6\delta$, then $\widehat{d}(z, q) \geq \widehat{d}(z, p')$, which contradicts our choice of q as a closest point on γ to z . Therefore $\widehat{d}(p, q) \leq 3K + 6\delta$, as required. \square

Recall that two points x and y define a halfspace $H(x, y)$. We now show that the image of the halfspace under the nearest point projection to a geodesic $[x, y]$ between x and y is contained in a

bounded neighbourhood of the half-segment of $[x, y]$ closest to y . As a partial converse, we show that if the nearest point projection of z lies sufficiently close to y , then $z \in H(x, y)$.

Proposition 3.7. *Let $z \in H(x, y)$, and let p be the nearest point to z on a geodesic $[x, y]$. Then $\widehat{d}(y, p) \leq \frac{1}{2}\widehat{d}(x, y) + 3\delta$. Conversely, if $\widehat{d}(y, p) \leq \frac{1}{2}\widehat{d}(x, y) - 3\delta$, then $z \in H(x, y)$.*

Proof. Let z be a point in the halfspace $H(x, y)$, let $[x, y]$ be a geodesic from x to y , and let p be a closest point on $[x, y]$ to z . This is illustrated below in Figure 6.

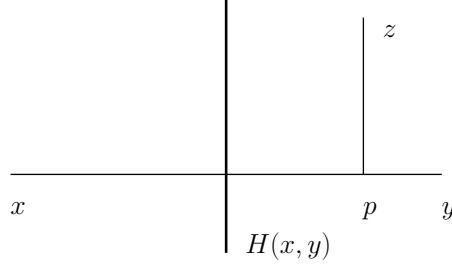


Figure 6: A point z in the halfspace $H(x, y)$.

The point z lies in the halfspace $H(x, y)$, so $\widehat{d}(z, y) \leq \widehat{d}(z, x)$. By the triangle inequality, the distance from z to x is less than or equal to the distance from z to x via p , which implies

$$\widehat{d}(z, y) \leq \widehat{d}(z, p) + \widehat{d}(p, x).$$

By Proposition 3.2, the path from z to y via p is close to being a geodesic, i.e. $\widehat{d}(z, p) + \widehat{d}(p, y) - 6\delta \leq \widehat{d}(z, y)$.

$$\widehat{d}(z, p) + \widehat{d}(p, y) - 6\delta \leq \widehat{d}(z, p) + \widehat{d}(p, x)$$

We may subtract $\widehat{d}(z, p)$ from both sides. As p lies on the geodesic $[x, y]$, the distance from x to y is equal to the distance from x to p plus the distance from p to y , i.e. $\widehat{d}(x, y) = \widehat{d}(x, p) + \widehat{d}(p, y)$. This gives the required inequality, $\widehat{d}(p, y) \leq \frac{1}{2}\widehat{d}(x, y) + 3\delta$.

For the converse, note that if z does not lie in the halfspace $H(x, y)$, then z lies in the halfspace $H(y, x)$, so $\widehat{d}(p, x) \leq \frac{1}{2}\widehat{d}(x, y) + 3\delta$. The point p lies on the geodesic $[x, y]$, so $\widehat{d}(p, x) = \widehat{d}(x, y) - \widehat{d}(p, y)$, which implies $\frac{1}{2}\widehat{d}(x, y) \leq \widehat{d}(p, y) + 3\delta$. Therefore, if $\widehat{d}(y, p) \leq \frac{1}{2}\widehat{d}(x, y) - 3\delta$ then this implies $z \in H(x, y)$, as required. \square

We have shown that the image of the nearest point projection of a halfspace $H(x, y)$ to $[x, y]$ is close to being half of this geodesic segment. If a geodesic segment $[x, z]$ fellow travels with $[x, y]$ for a sufficiently large initial segment, then we can estimate the image of the nearest point projection of $H(x, y)$ onto $[x, z]$. This will be the case as long as the nearest point projection of z to $[x, y]$ is sufficiently far from x . We now make this precise in the following proposition.

Proposition 3.8. *There are constants K_2 and K_3 , which only depend on δ , such that if z has nearest point projection p to a geodesic $[x, y]$, and $\widehat{d}(p, x) \geq \frac{1}{2}\widehat{d}(x, y) + K_2$, then the nearest point projection of $H(y, x)$ to $[x, z]$ is distance at most $\frac{1}{2}\widehat{d}(x, y) + K_3$ from x .*

Proof. We shall choose K_2 to be 27δ and K_3 to be 18δ . Let $H(y, x)$ be the halfspace defined by the pair of points x and y , and let $[x, y]$ be a geodesic from x to y . Let p be a nearest point to z on $[x, y]$. Let $a \in H(y, x)$, and let q be the closest point on $[x, y]$ to a . This is illustrated in Figure 7 below.

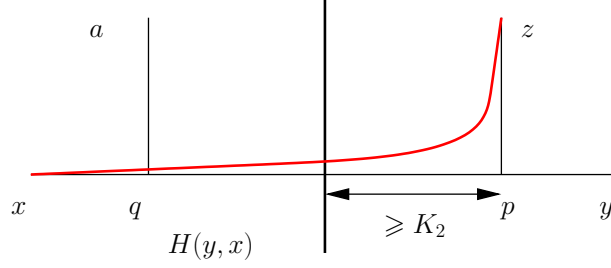


Figure 7: Closest point projection of $H(y, x)$ onto $[x, z]$.

By Proposition 3.7, nearest point projection maps the halfspace $H(y, x)$ to roughly half the geodesic $[x, y]$. This implies that $\widehat{d}(q, x) \leq \frac{1}{2}\widehat{d}(x, y) + 3\delta$, so $\widehat{d}(q, p) \geq K_2 - 3\delta$. Let t be a point on $[p, z]$, then as $K_2 \geq 17\delta$, Proposition 3.4 implies that $\widehat{d}(a, t) \geq \widehat{d}(a, q) + \widehat{d}(q, p) + \widehat{d}(p, t) - 24\delta$. As we chose K_2 to be 27δ , this implies $\widehat{d}(p, q) \geq 24\delta$, and so the closest point to a on $[x, p] \cup [p, z]$ lies on $[x, p]$. As $[x, p] \subset [x, y]$, Proposition 3.7 implies that the nearest point projection of $H(y, x)$ to $[x, p] \cup [p, z]$ is distance at most $\frac{1}{2}\widehat{d}(x, y) + 3\delta$ from x .

Let r be the nearest point to a on the geodesic $[x, z]$, and let s be the nearest point to a on the path $[x, p] \cup [p, z]$. By Proposition 3.2, the path from x to z via p is contained in a 3δ -neighbourhood of a geodesic $[x, z]$, so as nearest point projections to close paths are close, Proposition 3.6, this implies $\widehat{d}(r, s) \leq 3(3\delta) + 6\delta = 15\delta$. Therefore the nearest point projection of $H(y, x)$ to $[x, z]$ is distance at most $\frac{1}{2}\widehat{d}(x, y) + 18\delta$ from x , and so we may choose K_3 to be 18δ . \square

We now show that if $H(1, x)$ and $H(1, y)$ are two halfspaces far from the origin, such that the geodesic from $[x, y]$ passes close to the origin, then any other halfspace far from the origin hits at most one of $H(1, x)$ or $H(1, y)$.

Proposition 3.9. *There are constants K_4 and K_5 , which only depend on δ , such that for any geodesic $[x, y]$ with $\widehat{d}(1, x)$ and $\widehat{d}(1, y)$ at least $2\widehat{d}(1, [x, y]) + K_4$, and for any point z with $\widehat{d}(1, z) \geq 2\widehat{d}(1, [x, y]) + K_5$, the halfspace $H(1, z)$ hits at most one of the halfspaces $H(1, x)$ and $H(1, y)$.*

We remark that the hypotheses may be restated in terms of the Gromov product, $(x|y) = \frac{1}{2}(\widehat{d}(1, x) + \widehat{d}(1, y) - \widehat{d}(x, y))$, as in a δ -hyperbolic space the distance from the basepoint 1 to a geodesic $[x, y]$ is coarsely equivalent to the Gromov product of x and y .

Proof. We shall choose K_4 to be $2K_2 + K_3 + 42\delta$, and K_5 to be 24δ . Let $[x, y]$ be a geodesic from x to y , and let p be a closest point on $[x, y]$ to 1. It will be convenient to use the following form of our assumption that the distance from 1 to x is larger than twice the distance from 1 to p .

$$\frac{1}{2}\widehat{d}(1, x) \geq \widehat{d}(1, p) + \frac{1}{2}K_4 \quad (3.5)$$

Let r be the nearest point on $[1, x]$ to y , as illustrated below in Figure 8. We start by showing that r is a bounded distance from p . By Proposition 3.2, the path from y to 1 via p is almost a geodesic, so if t is a point on $[1, p]$ then $\widehat{d}(t, y) \geq \widehat{d}(y, p) + \widehat{d}(p, t) - 6\delta$. This implies that the closest point projection of y to $[x, p] \cup [p, 1]$ is distance at most 6δ from p . As the path $[1, p] \cup [p, x]$ lies in a 3δ -neighbourhood of the geodesic $[1, x]$, as nearest point projections to close paths are close, Proposition 3.6, this implies that

$$\widehat{d}(r, p) \leq 3(3\delta) + 6\delta + 6\delta = 21\delta. \quad (3.6)$$

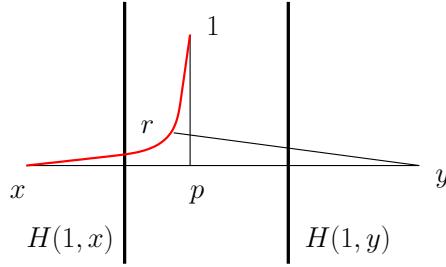


Figure 8: Disjoint halfspaces.

In order to use Proposition 3.8 to estimate the size of the nearest point projection of the halfspace $H(1, x)$ to the geodesic $[x, y]$ we need to show that r is closer to 1 than x . As r lies on a geodesic from 1 to x ,

$$\widehat{d}(x, r) = \widehat{d}(1, x) - \widehat{d}(1, r).$$

Using our initial assumption, line (3.5), we may rewrite this as

$$\widehat{d}(x, r) \geq \frac{1}{2}\widehat{d}(1, x) + \widehat{d}(1, p) - \widehat{d}(1, r) + \frac{1}{2}K_4.$$

Using the bound for $\widehat{d}(r, p)$ from line (3.6) and the triangle inequality, we obtain

$$\widehat{d}(x, r) \geq \frac{1}{2}\widehat{d}(1, x) + \frac{1}{2}K_4 - 21\delta.$$

Therefore we may apply Proposition 3.8, as we chose $K_4 \geq 2K_2 + 42\delta$. This implies that the nearest point projection of $H(1, x)$ to $[x, y]$ is contained in a $(\frac{1}{2}\widehat{d}(1, x) + K_3)$ -neighbourhood of x .

We now show that the nearest point projection of $H(1, x)$ to $[x, y]$ is a definite distance away from p . Let s be a point in the nearest point projection of $H(1, x)$ to $[x, y]$, so we have just shown that $\widehat{d}(s, x) \leq \frac{1}{2}\widehat{d}(1, x) + K_3$. As s lies on the geodesic $[x, p]$, this implies

$$\widehat{d}(s, p) \geq \widehat{d}(x, p) - \frac{1}{2}\widehat{d}(1, x) - K_3.$$

Using the triangle inequality applied to going from 1 to x via p , we obtain

$$\widehat{d}(s, p) \geq \widehat{d}(1, x) - \widehat{d}(1, p) - \frac{1}{2}\widehat{d}(1, x) - K_3.$$

Our initial assumption (3.5) now implies that $\widehat{d}(s, p) \geq K_4 - K_3$, which is greater than zero if $K_4 > K_3$. In fact $\widehat{d}(s, p) \geq 14\delta$, as we chose K_4 to be $2K_2 + K_3 + 42\delta$.

Our hypotheses are symmetric in x and y , so this also implies identical results for the image of the nearest point projection of $H(1, y)$ to $[x, y]$. In particular, this implies that the nearest point projections of $H(1, x)$ and $H(1, y)$ to the geodesic $[x, y]$ are disjoint.

We now consider a halfspace $H(1, z)$ which intersects both $H(1, x)$ and $H(1, y)$, and show there is an upper bound on $\widehat{d}(1, z)$. Assume that $H(1, z)$ intersects both $H(1, x)$ and $H(1, y)$. Let q be a closest point on $[x, y]$ to z . Up to relabelling x and y , we may assume that $q \in [p, y]$. This is illustrated in Figure 9 below.

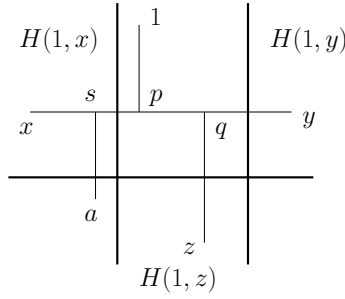


Figure 9: Three halfspaces.

Let a be a point lying in the intersection of $H(1, x)$ and $H(1, z)$, so in particular

$$\widehat{d}(a, z) \leq \widehat{d}(1, a). \quad (3.7)$$

Let s be a nearest point on $[x, y]$ to a . We showed above that $\widehat{d}(s, p) > 14\delta$, and therefore $\widehat{d}(s, q) > 14\delta$. As s and q are the nearest point projections of a and z onto the geodesic $[x, y]$, and $\widehat{d}(s, q) > 14\delta$, Proposition 3.4 implies that the path from a to z via s and q is close to being a geodesic, i.e. $\widehat{d}(a, z) \geq \widehat{d}(a, s) + \widehat{d}(s, p) + \widehat{d}(p, q) + \widehat{d}(q, z) - 24\delta$. Together with line (3.7), this implies

$$\widehat{d}(a, s) + \widehat{d}(s, p) + \widehat{d}(p, q) + \widehat{d}(q, z) - 24\delta \leq \widehat{d}(1, a).$$

By the triangle inequality, $\widehat{d}(1, a) \leq \widehat{d}(1, p) + \widehat{d}(p, s) + \widehat{d}(s, a)$, which gives

$$\widehat{d}(a, s) + \widehat{d}(s, p) + \widehat{d}(p, q) + \widehat{d}(q, z) - 24\delta \leq \widehat{d}(1, p) + \widehat{d}(p, s) + \widehat{d}(s, a).$$

Adding $\widehat{d}(1, p) - \widehat{d}(a, s) - \widehat{d}(s, p)$ to both sides, we obtain

$$\widehat{d}(1, p) + \widehat{d}(p, q) + \widehat{d}(q, z) \leq 2\widehat{d}(1, p) + 24\delta.$$

Using the triangle inequality, this shows that $\widehat{d}(1, z) \leq 2\widehat{d}(1, p) + 24\delta$. So we may choose K_5 to be 24δ . \square

We now show that if $H(1, x)$ is a halfspace far from the origin, then it is contained in a halfspace $H(1, y)$, with y a bounded distance closer to the origin, and we find an explicit lower bound on the distance between $H(1, x)$ and $H(y, 1)$. Furthermore, we show that for any point a in $H(y, 1)$, there is a point b such that the halfspace $H(a, b)$ contains $H(1, x)$, with an explicit lower bound on $\widehat{d}(a, b)$. Similarly, we show that for any point b in $H(1, x)$, there is a point a such that $H(y, 1) \subset H(b, a)$, again with an explicit lower bound on $\widehat{d}(a, b)$.

Proposition 3.10. *There are constants K_6 and K_7 , which only depend on δ , such that for any positive number A , and any point x with $\widehat{d}(1, x) \geq K_6 + 4A$, there is a point y on $[1, x]$ with $\widehat{d}(1, y) = \widehat{d}(1, x) - 2K_7 - 2A$, such that $H(1, x) \subset H(1, y)$. Furthermore, for any point a in $H(y, 1)$ there is a point b in $H(1, x)$, with $\widehat{d}(a, b) \geq 2A$, and $H(1, x) \subset H(a, b)$. Also, for any point b in $H(1, x)$, there is a point a in $H(y, 1)$ with $\widehat{d}(a, b) \geq 2A$, and $H(y, 1) \subset H(b, a)$.*

Proof. We shall choose K_7 to be $98\delta + 2K_1$ and K_6 to be $2K_7 + 6\delta$. We first show that $H(1, x) \subset H(1, y)$. Let p be a point in the nearest point projection of $H(y, 1)$ to $[1, x]$, and let q be a point in the nearest point projection of $H(1, x)$ to $[1, x]$. Applying Proposition 3.7 to the halfspace $H(1, y)$ gives

$$\widehat{d}(1, p) \leq \frac{1}{2}\widehat{d}(1, y) + 3\delta.$$

Using our assumption that $\widehat{d}(1, y) = \widehat{d}(1, x) - 2K_7 - 2A$ implies

$$\widehat{d}(1, p) \leq \frac{1}{2}\widehat{d}(1, x) + 3\delta - A - K_7. \quad (3.8)$$

Now applying Proposition 3.7 to the halfspace $H(1, x)$, we obtain

$$\widehat{d}(q, 1) \geq \frac{1}{2}\widehat{d}(1, x) - 3\delta. \quad (3.9)$$

Comparing (3.8) and (3.9) shows that if $K_7 \geq 13\delta$, then $\widehat{d}(p, q) \geq 7\delta$, and so $H(1, x)$ and $H(y, 1)$ are disjoint, which implies $H(1, x) \subset H(1, y)$, as required.

We now prove the second statement in Proposition 3.10.

Claim 3.11. For any point a in $H(y, 1)$ there is a point b in $H(1, x)$, with $\widehat{d}(a, b) \geq 2A$, and $H(1, x) \subset H(a, b)$.

Proof. Given a point a in $H(y, 1)$ we will choose a point b in the geodesic $[1, x]$, sufficiently far from a , and then show that $H(1, x) \subset H(a, b)$ by showing that the nearest point projections of $H(1, x)$ and $H(b, a)$ to $[1, x]$ are sufficiently far apart.

Let a be a point in the halfspace $H(y, 1)$, let p be a nearest point projection of a to $[1, x]$ and let b be a point on the geodesic $[1, x]$ distance $\widehat{d}(1, p) + K_7$ from x . This is illustrated below in Figure 10.

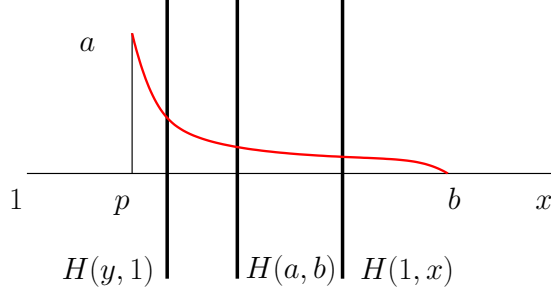


Figure 10: The halfspace $H(1, x)$ is contained in the halfspace $H(a, b)$.

We first find a lower bound for the distance between a and b . Using Proposition 3.2, which says that the path $[a, p] \cup [p, b]$ is contained in a 3δ -neighbourhood of $[a, b]$, and the triangle inequality, we obtain

$$\widehat{d}(a, b) \geq \widehat{d}(p, b) - 6\delta. \quad (3.10)$$

Let z be the midpoint of $[1, x]$. Then $\widehat{d}(p, z) \geq K_7 + A - 3\delta$, by Proposition 3.7, and our assumption on the distance between the halfspaces $H(1, x)$ and $H(1, y)$. Our choice of b then implies $\widehat{d}(z, b) \geq A - 3\delta$. Therefore $\widehat{d}(p, b) \geq K_7 + 2A - 12\delta$, and this is at least $2A$, as required, as we chose $K_7 \geq 12\delta$.

We now show that the halfspace $H(1, x)$ is contained in the halfspace $H(a, b)$. As nearest point projection is coarsely well-defined, Proposition 3.3, it suffices to show that the nearest point projections of $H(1, x)$ and $H(b, a)$ to $[1, x]$ are distance at least 7δ apart. By Proposition 3.7, if r is a point in the nearest point projection of $H(1, x)$ to $[1, x]$ then

$$\widehat{d}(r, x) \leq \frac{1}{2}\widehat{d}(1, x) + 3\delta. \quad (3.11)$$

Similarly, the nearest point projection of $H(b, a)$ to $[a, b]$ is distance at least $\frac{1}{2}\widehat{d}(a, b) - 3\delta$ from b . Let s be a closest point on $[a, b]$ to p . By Proposition 3.5, if r is a point in the nearest point projection of $H(b, a)$ to the subgeodesic $[s, b]$ of $[a, b]$, then

$$\widehat{d}(r, b) \geq \frac{1}{2}\widehat{d}(a, b) - 3\delta - K_1.$$

By Proposition 3.2, $\widehat{d}(s, p) \leq 3\delta$, and the path $[s, p] \cup [p, b]$ is contained in a 3δ -neighbourhood of $[s, b]$. Therefore, as nearest point projections to close paths are close, Proposition 3.6, this implies that if r is a nearest point in $[s, p] \cup [p, b]$ to $H(b, a)$ then

$$\widehat{d}(r, b) \geq \frac{1}{2}\widehat{d}(a, b) - 18\delta - K_1.$$

Nearest point projection onto the path $[s, p] \cup [p, b]$ is the same as nearest point projection onto the path $[p, s] \cup [s, p] \cup [p, b]$, and this latter path is contained in a 3δ -neighbourhood of $[p, b]$, so again applying Proposition 3.6, if r lies in the nearest point projection of $H(b, a)$ to $[p, b]$ then

$$\widehat{d}(r, b) \geq \frac{1}{2}\widehat{d}(a, b) - 33\delta - K_1.$$

Using (3.10), and the fact that $[p, b]$ is a geodesic subsegment of $[1, x]$, we obtain

$$\widehat{d}(r, b) \geq \frac{1}{2}\widehat{d}(1, x) - \frac{1}{2}\widehat{d}(1, p) - \frac{1}{2}\widehat{d}(b, x) - 36\delta - K_1.$$

The points r and b lie on the geodesic $[1, x]$, and $\widehat{d}(b, x) = \widehat{d}(1, p) + K_7$ from x , so this implies

$$\widehat{d}(r, x) \geq \frac{1}{2}\widehat{d}(1, x) + \frac{1}{2}K_7 - 36\delta - K_1. \quad (3.12)$$

Comparing (3.11) with (3.12) shows that the distance between the projections of the two halfspaces to $[1, x]$ is at least 7δ , as $K_7 \geq 92\delta + 2K_1$. Therefore the nearest point projections of $H(1, x)$ and $H(b, a)$ are sufficiently far apart, and so $H(1, x) \subset H(b, a)$, as required. \square

We now prove the final statement, by an analogous argument to the one above, though unfortunately not exactly the same, as the picture is not completely symmetric.

Claim 3.12. For any point b in $H(1, x)$, there is a point a in $H(y, 1)$ with $\widehat{d}(a, b) \geq 2A$, and $H(y, 1) \subset H(b, a)$.

Proof. Let b be a point in the halfspace $H(1, x)$, and let p be a nearest point projection of b to $[1, x]$. Let a be a point on the geodesic $[1, x]$ distance $\widehat{d}(p, x) - 2A - K_7$ from 1, and this is greater than zero, as $\widehat{d}(1, x) \geq 4A + 2K_7 + 6\delta$. This is illustrated below in Figure 11.

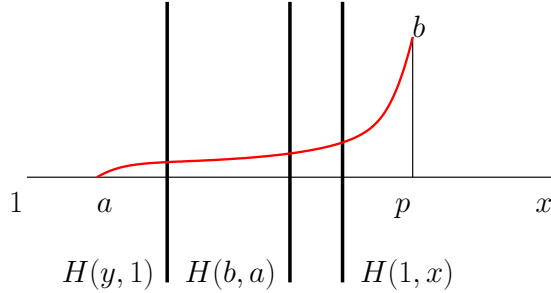


Figure 11: The halfspace $H(y, 1)$ is contained in the halfspace $H(b, a)$.

We start by showing that $\widehat{d}(a, b) \geq 2A$. Using Proposition 3.2, which says that the path $[a, p] \cup [p, b]$ is contained in a 3δ -neighbourhood of $[a, b]$, and the triangle inequality, we obtain

$$\widehat{d}(a, b) \geq \widehat{d}(a, p) - 6\delta. \quad (3.13)$$

Proposition 3.7, $\widehat{d}(p, x) \leq \frac{1}{2}\widehat{d}(1, x) + 3\delta$, and our choice of a , implies that $\widehat{d}(1, a) \leq \frac{1}{2}\widehat{d}(1, x) + 3\delta - 2A - K_7$. Therefore $\widehat{d}(a, p) \geq 2A + K_7 - 6\delta$, and so $\widehat{d}(a, b) \geq 2A$ as $K_7 \geq 12\delta$.

We wish to show that $H(y, 1) \subset H(b, a)$, and as before it suffices to show that the nearest point projections of the halfspaces $H(y, 1)$ and $H(b, a)$ to the geodesic $[1, x]$ are distance at least 7δ apart, as nearest point projection is coarsely well defined, Proposition 3.3. Let r be a point in the nearest point projection of $H(y, 1)$ to $[1, x]$ which is furthest from 1. By Proposition 3.7,

$\widehat{d}(1, r) \leq \frac{1}{2}\widehat{d}(1, y) + 3\delta$. Also, as $\widehat{d}(H(y, 1), H(1, x)) \geq A + K_7$, again using Proposition 3.7, this implies $\widehat{d}(x, y) \geq 2A + 2K_7 + 6\delta$. Therefore

$$\widehat{d}(1, r) \leq \frac{1}{2}\widehat{d}(1, x) - A - K_7 + 6\delta. \quad (3.14)$$

Now applying Proposition 3.7 to the halfspace $H(a, b)$, implies that if r is a point in the nearest point projection of $H(a, b)$ to $[a, b]$, then

$$\widehat{d}(a, r) \geq \frac{1}{2}\widehat{d}(a, b) - 3\delta.$$

Let s be a closest point on $[a, b]$ to p . As $[a, s]$ is a subgeodesic of $[a, b]$, we may use Proposition 3.5, which implies that if r is a point in the nearest point projection of $H(a, b)$ to $[a, s]$, then

$$\widehat{d}(a, r) \geq \frac{1}{2}\widehat{d}(a, b) - 3\delta - K_1.$$

By Proposition 3.2, $\widehat{d}(s, p) \leq 3\delta$, and the path $[a, p] \cup [p, s]$ is contained in a 3δ -neighbourhood of $[a, s]$. Therefore, as nearest point projections to close paths are close, Proposition 3.6, this implies that if r is a nearest point in $[a, p] \cup [p, s]$ to $H(a, b)$ then

$$\widehat{d}(a, r) \geq \frac{1}{2}\widehat{d}(a, b) - 18\delta - K_1.$$

Nearest point projection onto the path $[a, p] \cup [p, s]$ is the same as nearest point projection onto the path $[a, p] \cup [p, s] \cup [s, p]$, and this latter path is contained in a 3δ -neighbourhood of $[a, p]$, so again applying Proposition 3.6, if r lies in the nearest point projection of $H(a, b)$ to $[a, p]$ then

$$\widehat{d}(a, r) \geq \frac{1}{2}\widehat{d}(a, b) - 33\delta - K_1.$$

Using (3.13), and the fact that $[a, p]$ is a geodesic subsegment of $[1, x]$, we obtain

$$\widehat{d}(1, r) \geq \frac{1}{2}\widehat{d}(1, p) + \frac{1}{2}\widehat{d}(1, a) - 36\delta - K_1.$$

As we chose a such that $\widehat{d}(1, a) = \widehat{d}(p, x) - 2A - K_7$, this shows

$$\widehat{d}(1, r) \geq \frac{1}{2}\widehat{d}(1, x) - A - \frac{1}{2}K_7 - 36\delta - K_1. \quad (3.15)$$

Comparing (3.15) with (3.14) shows that the distance between the projections of the two halfspaces to $[1, x]$ is at least 7δ , as $K_7 \geq 98\delta + 2K_1$. Therefore the nearest point projections of $H(y, 1)$ and $H(a, b)$ are sufficiently far apart, and so $H(y, 1) \subset H(b, a)$, as required. \square

This completes the proof of Proposition 3.10. \square

Finally, we prove a result about a slightly more general definition of halfspaces. Let $H(1, x; C) = \{y \mid \widehat{d}(y, x) \leq \widehat{d}(1, x) + C\}$. We allow C to be negative, and if C is zero, this recovers the standard definition of a halfspace, and we shall continue to write $H(1, x)$ to mean $H(1, x; 0)$. We now show that a coarse halfspace is contained in a halfspace.

Proposition 3.13. *The halfspace $H(1, x; C)$ is contained in the halfspace $H(1, y)$, for $\widehat{d}(1, y) = \widehat{d}(1, x) - C - 9\delta$, if $0 \leq C \leq \widehat{d}(1, x) - 9\delta$.*

Proof. Let $z \in H(1, x; C)$, and let p be a nearest point to z in $[1, x]$, so $\widehat{d}(z, x) \leq \widehat{d}(z, 1) + C$. By the triangle inequality, $\widehat{d}(1, z) \leq \widehat{d}(1, p) + \widehat{d}(p, z)$, therefore

$$\widehat{d}(z, x) \leq \widehat{d}(1, p) + \widehat{d}(p, z) + C.$$

The path from z to x via p is almost a geodesic, Proposition 3.2, so $\widehat{d}(z, x) \geq \widehat{d}(z, p) + \widehat{d}(p, x) - 6\delta$. This implies

$$\widehat{d}(p, x) \leq \widehat{d}(1, p) + C + 6\delta.$$

As p lies on the geodesic $[1, x]$, we may rewrite this as

$$\widehat{d}(p, x) \leq \frac{1}{2}\widehat{d}(1, x) + \frac{1}{2}C + 3\delta.$$

By Proposition 3.7, this implies that z is contained in the halfspace $H(1, y)$, for $\widehat{d}(1, y) = \widehat{d}(1, x) - C - 9\delta$, as required. \square

4 Non-elementary semi-groups

We do not assume that our random walk is symmetric, so it will be convenient to know that if the group generated by the support of μ is non-elementary, then the semi-group generated by the support of μ contains a pair of independent pseudo-Anosov elements. This follows from well-known results on the structure of subgroups of the mapping class group, which we now briefly review. We will use the definitions and results of Ivanov [5], although the results we obtain could also be deduced from work of McCarthy [13] and Birman, Lubotzky and McCarthy [1].

We say an element h of the mapping class group is *pure*, if there is a disjoint collection of simple closed curves $\sigma(h)$ which are fixed individually by h , such that each complementary component of $\sigma(h)$ is also preserved, and furthermore h acts on each complementary component as either a pseudo-Anosov element or the identity. If the collection of simple closed curves $\sigma(h)$ has the property that no simple closed curve with non-zero intersection number with $\sigma(h)$ is fixed by h , then $\sigma(h)$ is called a *canonical reduction system* for h . If h is not pure, then we define the canonical reduction set $\sigma(h)$ to be the canonical reduction set of some pure power of h . Note that $\sigma(fgf^{-1}) = f\sigma(g)$. The canonical reduction system of a periodic or pseudo-Anosov element of the mapping class group is empty. Ivanov shows that given reducible elements of the mapping class group f and g , there is a product of f and g whose canonical reduction set is the intersection of the canonical reduction sets of f and g .

Lemma 4.1. [5, Lemma 5.1] *Let f and g be reducible elements of the mapping class group. Then there are positive numbers k and l , such that the canonical reduction set of $f^k g^l$ is the intersection of the canonical reduction sets for f and g .*

A pseudo-Anosov element f acts on \mathcal{PMF} with simple dynamics, there are a pair of fixed points, which we shall refer to as the stable fixed point λ_f^+ and the unstable fixed point λ_f^- . The stable

fixed point is attracting, i.e. every point in $\mathcal{PMF} \setminus \lambda_f^-$ converges to λ_f^+ under iteration by f , and the unstable fixed point is repelling, i.e. every point in $\mathcal{PMF} \setminus \lambda_f^+$ converges to λ_f^- under iteration by f^{-1} . Ivanov [5] shows that a pair of pseudo-Anosov elements either have the same pair of fixed points, or they have disjoint fixed points.

Lemma 4.2. [5, Lemma 5.11] *If f and g are pseudo-Anosov elements, then either $\text{fix}(f) = \text{fix}(g)$, or $\text{fix}(f) \cap \text{fix}(g)$ is empty.*

We will also make use of the following observation from Ivanov [5].

Lemma 4.3. [5, Chapter 11, exercise 3(a)] *Let f be a pseudo-Anosov element, with fixed points λ_f^\pm . Let g be any element such that $g(\lambda_f^+) \neq \lambda_f^-$. Then $f^n g$ is pseudo-Anosov for all sufficiently large n .*

We now show that if a non-elementary subgroup M is generated by a semi-group M^+ , then we can find a pair of independent pseudo-Anosov elements in M^+ .

Lemma 4.4. *Let M^+ be a semi-group in the mapping class group which generates a non-elementary group M . Then M^+ contains a pair of independent pseudo-Anosov elements.*

Proof. The group M contains two independent pseudo-Anosov elements. We now show that we can find a pair of independent pseudo-Anosov elements which in fact lie in M^+ .

Suppose that M^+ does not contain a pair of independent pseudo-Anosov elements, but does contain at least one pseudo-Anosov element. By Lemma 4.2, if M^+ contains pseudo-Anosov elements, they must all have common endpoints. Let f be a pseudo-Anosov element of M^+ , with fixed points λ_f^+ and λ_f^- . We will now show that all periodic and reducible elements of M^+ preserve the fixed point of f .

Suppose g is a periodic element of M^+ . If g does not preserve the fixed points of f , then gfg^{-1} is a pseudo-Anosov element with distinct fixed points, and furthermore lies in M^+ , as g^{-1} may be written as a positive power of g . So all periodic elements of M^+ preserve the fixed points of f .

Suppose g is a reducible element of M^+ . At least one of g or g^2 has the property that the image of λ_f^+ is not λ_f^- , so possibly after replacing g by its square, we may assume that g has this property. Then by Lemma 4.3, the element $f^n g$ is pseudo-Anosov for all sufficiently large n . In particular $gf^n gg^{-1} = gf^n$ is pseudo-Anosov, and lies in M^+ . As we have assumed that M^+ contains no independent pseudo-Anosov elements, gf^n must have the same fixed points as f , but this implies that g preserves the fixed points of f .

We have shown that if M^+ contains pseudo-Anosov elements, then they must all have common fixed points, and all other elements of M^+ preserve the set of fixed points. But then the inverses of all elements of M^+ preserve the set of fixed points, so in fact the entire group M preserves the pair of fixed points, which contradicts the fact that M is non-elementary. So we may assume that the semi-group M^+ contains no pseudo-Anosov elements.

If there is a disjoint collection of essential simple closed curves in the surface Σ fixed by all elements of M^+ , then this disjoint collection of simple closed curves is also fixed by all inverses of elements in M^+ , and hence by the whole group M . As M is non-elementary it fixes no disjoint collection of simple closed curves in Σ , so this implies that the semi-group M^+ also fixes no disjoint collection of simple closed curves in the surface Σ . However, it could *a priori* be the case that all reducible elements of M^+ have common simple closed curves in their canonical reduction sets, but these are not preserved by elliptic elements of M^+ .

Let ρ be the intersection of the canonical reduction sets of all reducible elements of M^+ , i.e. $\rho = \bigcap \{\sigma(g) \mid g \in M^+, g \text{ reducible}\}$. Lemma 4.1 shows that given reducible elements of the mapping class group f and g , there is an element of the semi-group generated by f and g whose canonical reduction set is the intersection of the canonical reduction sets for f and g . This implies that there is a reducible element $r \in M^+$ with $\sigma(r) = \rho$.

Suppose $g \in M^+$ does not preserve ρ . Let k be the smallest positive integer such that g^k is pure. If g is periodic, then k is the order of g , and $\sigma(g)$ is empty. If g is reducible, then $\rho \subset \sigma(g)$. By Lemma 4.3, for all sufficiently large positive integers l , the element $r^l g^k$ is pseudo-Anosov on each component of $\Sigma \setminus \rho$ on which either r or g^k is pseudo-Anosov. In particular, $\sigma(r^l g^k) = \rho$, and $r^l g^k$ is not periodic. As we have assumed M^+ contains no pseudo-Anosov elements, $r^l g^k$ must in fact be reducible.

Consider conjugating $r^l g^k$ by g , i.e. $g f^l g^k g^{-1} = g f^l g^{k-1}$. This gives a reducible element of M^+ with canonical reduction set $\sigma(g f^l g^{k-1}) = g \sigma(f^l g^k) = g(\rho) \neq \rho$. So we can construct an element of M^+ with canonical reduction set $\rho \cap g(\rho)$, which is strictly smaller than ρ , which contradicts the fact that ρ is the intersection of all of the canonical reduction sets of all of the reducible elements of M^+ . So ρ is preserved by all elements of M^+ , and hence by the inverse of all elements of M^+ , and therefore by the entire group M . But then the group M is reducible, not non-elementary, a contradiction. \square

5 Exponential decay

In this section we show that the harmonic measure of halfspaces $H(1, x)$ decay exponentially with their distance from the origin. As the number of disjoint halfspaces $H(1, x)$ grows exponentially in $\widehat{d}(1, x)$, this is clearly true on average, but in this section we show that the measure of all halfspaces decays as $L^{\widehat{d}(1, x)}$, for some constant $L < 1$ which does not depend on the choice of halfspaces. We then show that there is a similar estimate for the convolutions measures μ_n , i.e. $\mu_n(H(1, x)) \leq Q L^{\widehat{d}(1, x)}$, for some constant Q , and where $L < 1$ is the same constant as for ν .

We will need to use the fact that the group generated by the support of the random walk is non-elementary, and it will be convenient to use the fact that non-elementary subgroups have free subgroups that act on the complex of curves in a similar manner to Schottky groups acting on hyperbolic space.

Definition 5.1. We say that a pair of elements $a, b \in G$ are a *Schottky pair* if there are disjoint halfspaces A^+, A^-, B^+ and B^- in the relative space \widehat{G} , such that $a(\widehat{G} \setminus A^-) \subset A^+$, $a^{-1}(\widehat{G} \setminus A^+) \subset A^-$, $b(\widehat{G} \setminus B^-) \subset B^+$ and $b^{-1}(\widehat{G} \setminus B^+) \subset B^-$. We will refer to a choice of halfspaces A^\pm, B^\pm , with the properties above, as Schottky halfspaces for the Schottky pair a, b . \diamond

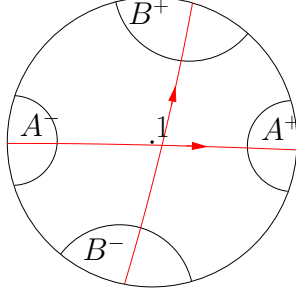


Figure 12: Halfspaces for a Schottky pair.

This is illustrated in Figure 12 above. The red lines correspond to axes for a and b . The subgroup generated by a Schottky pair is a free group with two generators, in which all non-identity elements are pseudo-Anosov. A non-elementary subgroup of the mapping class group contains a pair of pseudo-Anosov elements with distinct endpoints in \mathcal{PMF} , and these correspond to distinct points in the Gromov boundary of the complex of curves by work of Klarreich [10]. Therefore there are integers p and q such that a^p and b^q are a Schottky pair. Furthermore, we may replace p and q with any larger integers.

We first show that the harmonic measure of a Schottky halfspace is non-zero, for any Schottky pair contained in the support of μ .

Proposition 5.2. *Let A^+ be a Schottky halfspace, whose corresponding pseudo-Anosov element lies in the subgroup generated by the support of μ . Then $\nu(\overline{A^+}) > 0$.*

Proof. Suppose $\nu(\overline{A^+}) = 0$. The measure ν is μ_n -invariant, i.e. $\nu(X) = \sum \mu_n(g)\nu(g^{-1}X)$, so if $\mu_n(g) \neq 0$, then $\nu(g^{-1}\overline{A^+}) = 0$. As a lies in the support of μ , there is an n such that $\mu_n(a) > 0$. This implies that $\nu(a^{-k}\overline{A^+}) = 0$ for all k . However, the complement of the union of the $a^{-k}A^+$ is the unstable fixed point of a , which has measure zero as ν is non-atomic. This means that the total measure $\nu(\mathcal{PMF})$ is zero, a contradiction. \square

We now show that the harmonic measure of $H(1, x)$ is bounded away from 1, at least for $\widehat{d}(1, x)$ sufficiently large.

Proposition 5.3. *Let μ be a probability distribution on G whose support generates a non-elementary subgroup, and let ν be the corresponding harmonic measure. Then there are constants K_8 and $\epsilon > 0$ such that $\nu(\overline{H(1, x)}) \leq 1 - \epsilon$, for all x with $\widehat{d}(1, x) \geq K_8$. The constant K_8 only depends on the support of μ . The constant ϵ depends on μ .*

Proof. Let M be the subgroup generated by the support of μ . As this subgroup is non-elementary, it contains a Schottky pair, $\{a, b\}$. As a and b are pseudo-Anosov, the sequences a^k and b^k are quasi-geodesic, with distinct endpoints, so possibly after replacing a and b by positive powers a^r and b^s , we may assume that $\{H(1, a^{\pm 1}), H(1, b^{\pm 1})\}$ are a collection of Schottky halfspaces for $\{a, b\}$, and $\min\{\widehat{d}(1, a), \widehat{d}(1, b)\} > 2\widehat{d}(1, [a, b]) + K_4$. By Proposition 3.9 there is a $K_8 = 2\widehat{d}(1, [a, b]) + K_5$ such that any halfspace $H(1, x)$ with $\widehat{d}(1, x) \geq K_8$ intersects at most one of the Schottky halfspaces. In particular this means that the half space $H(1, x)$ is disjoint from at least three of the Schottky half

spaces, so we may choose ϵ to be the minimum of the sum of any three of the harmonic measures of the four Schottky halfspaces. \square

We now show that the harmonic measure of halfspaces $H(1, x)$ decays exponentially in $\widehat{d}(1, x)$ as x varies, at least for x with $\widehat{d}(1, x)$ sufficiently large.

Lemma 5.4. *Let μ be a probability distribution on G whose support is relatively bounded and which generates a non-elementary subgroup, and let ν be the corresponding harmonic measure. Then there are constants K_9 and $L < 1$, such that if $\widehat{d}(1, x) \geq K_9$ then $\nu(\overline{H(1, x)}) \leq L^{\widehat{d}(1, x)}$. Both K_9 and L depend on μ .*

Proof. As the subgroup generated by the support of μ is non-elementary, the subgroup contains a Schottky pair, so by Proposition 5.3, there are constants K_8 and ϵ such that $\nu(\overline{H(1, x)}) \leq 1 - \epsilon$, for $\widehat{d}(1, x) \geq K_8$.

Before considering more general random walks, we explain the argument in the case of the nearest neighbour random walk on the Cayley graph. Consider a pair of nested halfspaces $H(1, x_2)$ and $H(1, x_1)$, where $\widehat{d}(1, x_2) \geq 2K_8 + K_6$, and $x_1 \in [1, x_2]$ with $\widehat{d}(1, x_1) = \widehat{d}(1, x_2) - K_8 - 2K_7$. Given a halfspace $H(1, x_1)$, we will refer to the set of points which are equal distance from both 1 and x_1 as the *equidistant set* for $H(1, x_1)$, which we shall denote by $E(1, x_1)$. We remark that *a priori* it may be the case that equidistant set $E(1, x_1) = H(1, x_1) \cap H(x_1, 1)$ has measure strictly larger than zero. Any sample path that converges into the limit set of $H(1, x_2)$, must pass through the equidistant set of $H(1, x_1)$. So we can use the formula for conditional probability, conditioning on the location at which the sample path first hits the equidistant set of $H(1, x_1)$. By Proposition 3.10, for any point $a \in E(1, x_1)$, the halfspace $H(1, x_2)$ is contained in a halfspace $H(a, b)$, with $\widehat{d}(a, b) \geq K_8$, so the probability that you converge into the limit set of $H(1, x_2)$ from any point of $E(1, x_1)$ is at most $1 - \epsilon$, so, in particular, the probability that you converge into the limit set of $H(1, x_2)$ is at most $1 - \epsilon$. We may apply this argument to any nested sequence of halfspaces, each distance at least $2K_8 + K_6$ apart, so the probability that you converge into the limit set of a halfspace $H(1, x)$ with $\widehat{d}(1, x) \geq (2K_8 + K_6)n$ is most $(1 - \epsilon)^{n-1}$.

For more general probability distributions, we need to replace the condition of hitting the equidistant set of a halfspace, with the condition of hitting some suitably large neighbourhood of the equidistant set.

Let H_i be a sequence of halfspaces $H(1, x_i)$, where the x_i lie on a common geodesic starting at 1, with $\widehat{d}(1, x_{i+1}) \geq \widehat{d}(1, x_i) + 2K_8 + K_6$ for each i . Consider a sample path which converges to a boundary point λ in \overline{H}_k . Such a sample path must have infinitely many elements in H_1 . If such a sample path does not hit $H_1 \setminus H_k$ then it must first hit H_1 inside H_k , after a jump of distance at least $k(2K_8 + K_6)$. Pick k such that $k(2K_8 + K_6)$ is larger than the relative diameter of the support of μ .

For a sample path w , let $F(w)$ be the group element corresponding to location at which w first hits H_1 . This is well defined for sample paths which converge to $\lambda \in \overline{H}_2$. All sample paths which converge to $\lambda \in \overline{H}_k$ hit $H_1 \setminus H_k$, so conditioning on $F(w)$ gives

$$\mathbb{P}(\lambda(w) \in \overline{H}_{k+1} \mid \lambda(w) \in \overline{H}_k) \leq \sum_{g \in H_1 \setminus H_k} \mathbb{P}(F(w) = g) \nu(g^{-1} \overline{H}_{k+1})$$

By Proposition 3.10, for any point g in $H_1 \setminus H_k$, there is a point b in H_{k+1} such that $H_{k+1} \subset H(g, b)$, and $\widehat{d}(g, b) \geq K_8$. Therefore, by Proposition 5.3, $\nu(g^{-1}\overline{H_{k+1}}) \leq 1 - \epsilon$, so

$$\mathbb{P}(\lambda(\omega) \in \overline{H_{k+1}} \mid \lambda(\omega) \in \overline{H_k}) \leq 1 - \epsilon$$

This implies that $\nu(\overline{H_{k+1}}) \leq \nu(\overline{H_k})(1 - \epsilon)$, so by induction $\nu(\overline{H_{k+l}}) \leq (1 - \epsilon)^{l+1}$. Therefore $\nu(\overline{H_n}) \leq L^n$, for $\widehat{d}(1, x) \geq K_9$, where we may choose $L = (1 - \epsilon)^{1/(k+1)}$ and $K_9 = k(2K_8 + K_6)$. \square

The measure ν is the weak limit of the measures μ_n , so one may hope there is exponential decay for halfspaces for the μ_n -measures, at least for large n . In fact, we now show that there is an upper bound for $\mu_n(H(1, x))$ which decays exponentially in $\widehat{d}(1, x)$, for all n .

Lemma 5.5. *There are constants $K_{10}, L < 1$ and Q , which only depend on δ and μ , such that if $H(1, x)$ is a halfspace with $\widehat{d}(1, x) \geq K_{10}$, then $\mu_n(H(1, x)) \leq QL^{\widehat{d}(1, x)}$.*

Proof. We shall choose $K_{10} = K_6 + 2K_8 + K_9$. Let $H(1, x)$ be a halfspace with $\widehat{d}(1, x) \geq 2K_8 + K_6$. Then, by Proposition 3.10, we may choose $y \in [1, x]$ with $\widehat{d}(1, y) = \widehat{d}(1, x) - K_8 - 2K_7$, such that $H(1, x) \subset H(1, y)$, and, furthermore, for any $b \in H(1, x)$, there is a point a in $H(y, 1)$ such that the halfspace $H(y, 1)$ is contained in the halfspace $H(b, a)$, with $\widehat{d}(a, b) \geq K_8$. By Proposition 5.3, the harmonic measure of $\overline{H(y, 1)}$ viewed from any $b \in H(1, x)$ is at most $1 - \epsilon$, i.e. if ν_b is the harmonic measure induced by a random walk starting at b instead of 1, then $\nu_b(\overline{H(y, 1)}) \leq 1 - \epsilon$. At time n , the halfspace $H(1, x)$ has measure $\mu_n(H(1, x))$, so the proportion of these sample paths which converge into the limit set of $H(y, 1)$ is at most $1 - \epsilon$. Therefore at least ϵ of these sample paths converge into the complement of the limit set of $H(y, 1)$, which is contained in the limit set of $H(1, y)$, i.e. $\nu(\overline{H(1, y)}) \geq \epsilon \mu_n(H(1, x))$. This implies

$$\mu_n(H(1, x)) \leq \frac{1}{\epsilon} \nu(\overline{H(1, y)}).$$

We chose $\widehat{d}(1, x) \geq K_6 + 2K_8 + K_9$, and $\widehat{d}(1, y) = \widehat{d}(1, x) - K_8 - 2K_7$, so we may apply the exponential decay bounds from Lemma 5.4 to obtain

$$\mu_n(H(1, x)) \leq \frac{1}{\epsilon} L^{\widehat{d}(1, x) - K_8 - 2K_7}.$$

Therefore we may choose Q to be $\frac{1}{\epsilon} L^{-K_8 - 2K_7}$, and this depends on δ and μ , but not on x or n . Therefore we have shown that there is a constant Q such that $\mu_n(H(1, x)) \leq QL^{\widehat{d}(1, x)}$, for all n , as long as $\widehat{d}(1, x) \geq K_{10} = K_6 + 2K_8 + K_9$. The constant L may be chosen to have the same value as the constant L from Lemma 5.4. \square

We remark that for the nearest neighbour random walk, for small n and $\widehat{d}(1, x)$ large, $\mu_n(H(1, x))$ will be zero until n is at least $\widehat{d}(1, x)$, so $\mu_n(H(1, x))$ need not be monotonically decreasing in n .

6 Linear progress

We may now complete the proof of the main theorem.

Theorem 1.1. *Let G be the mapping class group of an orientable surface of finite type, which is not a sphere with three or fewer punctures, and consider the random walk generated by a probability distribution μ , whose support is bounded in the relative metric and which generates a non-elementary subgroup of the mapping class group, and which has finite first moment. Then there is a constant $\ell > 0$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \widehat{d}(1, w_n) = \ell$ almost surely.*

The fact that the limit $\frac{1}{n} \widehat{d}(1, w_n)$ exists almost surely with respect to \mathbb{P} follows from a standard application of Kingman's subadditive ergodic theorem. The main task of this section is to show that this limit is strictly greater than zero. We now state Kingman's subadditive ergodic theorem [9], using the version from Woess [14, Theorem 8.10].

Theorem 6.1. [9, 14] *Let (Ω, \mathbb{P}) be a probability space and $U : \Omega \rightarrow \Omega$ a measure preserving transformation. If W_n is a subadditive sequence of non-negative real-valued random variables on Ω , that is, $W_{n+k} \leq W_n + W_k \circ U^n$ for all $k, n \in \mathbb{N}$, and W_1 has finite first moment, then there is a U -invariant random variable W_∞ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} W_n = W_\infty$$

\mathbb{P} -almost surely, and in expectation.

We will choose (Ω, \mathbb{P}) to be the path space $(G^{\mathbb{Z}^+}, \mathbb{P})$ for the random walk determined by (G, μ) . We will choose U to be the Bernoulli shift in the space of increments of the random walk, which is an ergodic measure-preserving transformation on $(G^{\mathbb{Z}^+}, \mathbb{P})$. Set $W_n = \widehat{d}(1, w_n)$. Then $W_k \circ U^n = \widehat{d}(1, w_n^{-1} w_{n+k})$, which is the relative distance between w_n and w_{n+k} , so the triangle inequality implies that W_n is subadditive. We have assumed that μ has finite first moment with respect to the word metric on G , and as $\widehat{d}(1, g) \leq |g|$ for all $g \in G$, this implies that μ also has finite first moment with respect to the relative metric on G . Theorem 6.1 then implies that $\lim_{n \rightarrow \infty} \frac{1}{n} \widehat{d}(1, w_n) = \ell$ exists almost surely, and in expectation, and in fact is constant almost surely. As $\widehat{d}(1, w_n) \geq 0$ for all n , this implies that $\ell \geq 0$. It remains to show that ℓ is strictly larger than zero. As the limit is constant almost surely, and the limit exists in expectation, it suffices to show that the limit $\frac{1}{n} \mathbb{E}(\widehat{d}(1, w_n))$ is bounded away from zero. We will show that for m sufficiently large, the expected difference between $\widehat{d}(1, w_n)$ and $\widehat{d}(1, w_{n+m})$ is bounded away from zero.

Lemma 6.2. *There is a constant N , such that if $m > N$, then $\mathbb{E}(\widehat{d}(1, w_{n+m}) - \widehat{d}(1, w_n)) \geq \delta > 0$. The constant N depends on μ , but is independent of m and n .*

This suffices to prove Theorem 1.1, as we now explain. Consider

$$\mathbb{E}(\widehat{d}(1, w_{2N})) = \mathbb{E}(\widehat{d}(1, w_{2N}) - \widehat{d}(1, w_N) + \widehat{d}(1, w_N) - \widehat{d}(1, w_0))$$

This is equal to

$$= \mathbb{E}(\widehat{d}(1, w_{2N}) - \widehat{d}(1, w_N)) + \mathbb{E}(\widehat{d}(1, w_N)) \geq 2\delta$$

A similar argument shows $\mathbb{E}(\widehat{d}(1, w_{kN})) \geq k\delta$. As $\lim_{n \rightarrow \infty} \frac{1}{n} \widehat{d}(1, w_n) = \ell$ is constant almost surely, this shows $\ell \geq \delta/N > 0$, as required.

We now prove Lemma 6.2.

Proof. Consider doing a random walk of length n , followed by one of length m . We can compute the expected change in relative length from time n to time $n + m$, which we shall denote $\Delta_{n,m}$.

$$\Delta_{n,m} = \mathbb{E}(\widehat{d}(1, w_{n+m}) - \widehat{d}(1, w_n))$$

We may rewrite this using the definition of expected value, giving

$$\Delta_{n,m} = \sum_{x \in G} \mu_n(x) \sum_{y \in G} \mu_m(y) (\widehat{d}(1, xy) - \widehat{d}(1, x)).$$

The fact that μ has finite first moment means that this sum is absolutely convergent, so we can swap the order of summation. Furthermore, as $\widehat{d}(1, xy) = \widehat{d}(x^{-1}, y)$, and $\widehat{d}(1, x) = \widehat{d}(x^{-1}, 1)$ we can rewrite this as

$$\Delta_{n,m} = \sum_{y \in G} \mu_m(y) \sum_{x \in G} \mu_n(x) (\widehat{d}(x^{-1}, y) - \widehat{d}(x^{-1}, 1))$$

We may split the sum up into two parts, depending on whether or not $\widehat{d}(1, y) > A$, where A is a constant which depends on μ . We will choose $A \geq K_6 + 2K_8 + K_9 + 12\delta$, and furthermore, we will choose A to be sufficiently large such that $rL^r < \delta/2Q$, for all real numbers $r \geq A$, and this is possible as $L < 1$. Here the constants K_6, K_8, K_9, L and Q are those defined previously in Sections 3 and 5, though we emphasize that Q and L are the constants defined for the reflected random walk $(G, \tilde{\mu})$. We shall write \widehat{B}_A for the ball of radius A about 1 in \widehat{G} .

$$\Delta_{n,m} = \sum_{y \in \widehat{B}_A} \mu_m(y) \sum_{x \in G} \mu_n(x) (\widehat{d}(x^{-1}, y) - \widehat{d}(x^{-1}, 1)) \quad (6.1)$$

$$+ \sum_{y \in G \setminus \widehat{B}_A} \mu_m(y) \sum_{x \in G} \mu_n(x) (\widehat{d}(x^{-1}, y) - \widehat{d}(x^{-1}, 1)) \quad (6.2)$$

We now find lower bounds for lines (6.1) and (6.2) in turn. First, in line (6.1), $\widehat{d}(x^{-1}, y) - \widehat{d}(x^{-1}, 1) \geq -\widehat{d}(1, y)$, by the triangle inequality. We have also assumed that $\widehat{d}(1, y) \leq A$. This implies the following lower bound for (6.1).

$$(6.1) \geq -\mu_m(\widehat{B}_A)A \quad (6.3)$$

We now find a lower bound for line (6.2), and we will simplify the notation by replacing x with x^{-1} . This makes no difference to the sum as we are summing over all elements of G .

$$(6.2) \geq \sum_{y \in G \setminus \widehat{B}_A} \mu_m(y) \sum_{x \in G} \mu_n(x^{-1}) (\widehat{d}(x, y) - \widehat{d}(x, 1))$$

Let N be the collection of $x \in G$ for which $\widehat{d}(x, y) - \widehat{d}(x, 1) \leq 0$, i.e. N is the halfspace $H(1, y)$. Let P be the region on which $\widehat{d}(x, y) - \widehat{d}(x, 1) \geq 3\delta$, so P is the halfspace $H(y, 1; -3\delta)$, using the notation from Proposition 3.13. This is illustrated below in Figure 13.

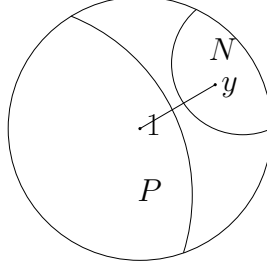


Figure 13: Positive and negative regions.

Let z be a point on $[1, y]$ distance 12δ from y . We now show that the region P contains the halfspace $H(z, 1)$. The complement of P is contained in $H(1, y; 3\delta)$, and by Proposition 3.13, the complement of P is contained in $H(1, z)$, for $\widehat{d}(1, z) = \widehat{d}(1, y) - 12\delta$. Therefore P contains the halfspace $H(z, 1)$. By the triangle inequality, $\widehat{d}(x, y) - \widehat{d}(x, 1) \geq -\widehat{d}(1, y)$ on N , and $\widehat{d}(x, y) - \widehat{d}(x, 1) \geq 3\delta$ on P . This gives the following lower bound for line (6.2).

$$(6.2) \geq \sum_{y \in G \setminus \widehat{B}_A} \mu_m(y) \left[-\widehat{d}(1, y) \widetilde{\mu}_n(H(1, y)) + 3\delta \widetilde{\mu}_n(H(z, 1)) \right]$$

Here $\widetilde{\mu}_n$ is the reflected random walk measure, i.e. $\widetilde{\mu}_n(x) = \mu_n(x^{-1})$. Using that fact that $\widetilde{\mu}_n(H(z, 1)) \geq 1 - \widetilde{\mu}_n(H(1, z))$, we obtain

$$(6.2) \geq \sum_{y \in G \setminus \widehat{B}_A} \mu_m(y) \left[-\widehat{d}(1, y) \widetilde{\mu}_n(H(1, y)) + 3\delta(1 - \widetilde{\mu}_n(H(1, z))) \right].$$

In line (6.2) we have assumed that $\widehat{d}(1, y) \geq 2K_8 + K_6 + 12\delta$, so the halfspaces $H(1, y)$ and $H(1, z)$ both satisfy the hypotheses of Lemma 5.5, so we may estimate $\widetilde{\mu}_n$ in terms of $\widetilde{\nu}$.

$$(6.2) \geq \sum_{y \in G \setminus \widehat{B}_A} \mu_m(y) \left[3\delta - \widehat{d}(1, y) Q \widetilde{\nu}(\overline{H(1, y)}) - \delta Q \widetilde{\nu}(\overline{H(z, 1)}) \right]$$

Here $\widetilde{\nu}$ is the harmonic measure corresponding to the reflected random walk defined by $(G, \widetilde{\mu})$, and Q is a constant which depends on $\widetilde{\mu}$, but not on y or z . In line (6.2) we have assumed that $\widehat{d}(1, y) \geq K_9 + 12\delta$, so we may apply Lemma 5.4, which says that the harmonic measure of halfspaces decays exponentially.

$$(6.2) \geq \sum_{y \in G \setminus \widehat{B}_A} \mu_m(y) \left[3\delta - Q \widehat{d}(1, y) L^{\widehat{d}(1, y)} - Q \delta L^{\widehat{d}(1, z)} \right]$$

Now using the facts that $\widehat{d}(1, y) \geq \delta$, and we chose A such that $rL^r \leq \delta/2Q$ for all $r \geq A$, this implies that the term in square brackets in the line above is at least 2δ . Therefore

$$(6.2) \geq (1 - \mu_m(\widehat{B}_A)) 2\delta. \tag{6.4}$$

Recall that $\Delta_{n,m} = (6.1) + (6.2)$, and now using the lower bounds from lines (6.3) and (6.4), we obtain

$$\Delta_{n,m} \geq 2\delta - \mu_m(\widehat{B}_A)(A + 2\delta).$$

The harmonic measure of a bounded set in \widehat{G} is zero, and so the random walk is transient on relatively bounded sets. In particular, for fixed A there is an N such that $\mu_m(\widehat{B}_A) < \delta/(A + 2\delta)$ for all $m \geq N$, and this will be our choice of N , which depends on μ . In particular, this means that $\Delta_{n,m} \geq \delta$, for all n , and for all $m \geq N$, and so $\Delta_{n,m}$ is bounded away from zero, as required. \square

This completes the proof of lemma 6.2, and hence of Theorem 1.1.

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